Integration (9)
Partial Fractions

Type I Rem: If degree of Nr is greater or equal to the degree of Dr. first divide then express in the form of Quotient + $\frac{\text{Rem}}{\text{Divisor}}$.

Type I $\int \frac{p(x)}{(x-a)(x-b)}\,dx$ where $\deg p(x) \geq \deg[(x-a)(x-b)]$

e.g. Q.1 $I = \int \frac{x^3}{(x-1)(x-2)}\,dx$

$$= \int \frac{x^3}{x^2-3x+2}\,dx$$

$$\begin{array}{r} x+3 \\ x^2-3x+2 \,\big)\; x^3 \qquad\qquad \\ \underline{\cancel{x^3} - 3x^2 + 2x} \\ - \quad + \quad - \\ 3x^2 - 2x \\ \underline{\cancel{3x^2} - 9x + 6} \\ - \quad + \quad - \\ 7x - 6 \end{array}$$

$$= \int \left[x+3+\frac{7x-6}{(x-1)(x-2)}\right] dx \quad \left(\text{Quo} + \frac{\text{Rem}}{\text{Div}}\right) \quad \text{—(1)}$$

Now $\frac{7x-6}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$ —(2)

$7x-6 = A(x-2) + B(x-1)$ [Multiply both sides by $(x-1)(x-2)$]

Put $x=1$

$7-6 = A(1-2) \Rightarrow A = -1$

Put $x=2$

$14-6 = B(2-1) \Rightarrow 8 = B$

Put in (2)

$$\frac{7x-6}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{8}{x-2}$$

Put in (1)

$$I = \int \left[x+3 - \frac{1}{x-1} + \frac{8}{x-2}\right] dx$$

$$= \frac{x^2}{2} + 3x - \log(x-1) + 8\log|x-2| + C \quad (2)$$

Type II: $\int \frac{P(x)}{(x-a)^n(x-b)}\,dx$ deg $P(x) <$ deg(Dr)

Reduce

$$\frac{P(x)}{(x-a)^n(x-b)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \dots + \frac{A_n}{(x-a)^n} + \frac{B}{x-b}$$

Q. 2 $I = \int \frac{3x+1}{(x-2)^2(x+2)}\,dx$

$$\frac{3x+1}{(x-2)^2(x+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} \quad -(1)$$

Multiply both sides by $(x-2)^2(x+2)$

$$3x+1 = A(x-2)(x+2) + B(x+2) + C(x-2)^2 \quad -(2)$$

Put $x = 2$

$$6+1 = B(2+2) \Rightarrow B = 7/4$$

Put $x = -2$

$$-6+1 = C(-2-2)^2 \Rightarrow C = \frac{-5}{16}$$

Comparing coeff of x^2 on both sides of (2)

$$0 = A + C \Rightarrow C = -A \Rightarrow A = 5/16$$

$$\therefore I = \int \left[\frac{5}{16(x-2)} + \frac{7}{4(x-2)^2} - \frac{5}{16(x+2)}\right] dx \quad \text{from (1)}$$

$$= \frac{5}{16}\log|x-2| + \frac{7}{4}\frac{(x-2)^{-1}}{-1} - \frac{5}{16}\log|x+2| + C$$

$$= \frac{5}{16}\log|x-2| - \frac{7}{4(x-2)} - \frac{5}{16}\log|x+2| + C$$

Type III $\int \frac{P(x)}{(ax^2+b)(x+c)}\,dx$ $\quad \deg P(x) < \deg (Dr)$

Q.3 $I = \int \frac{x}{(x-1)(x^2+4)}\,dx$

$$\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} \quad -(1)$$

Multiply both sides by $(x-1)(x^2+4)$ $\quad -(2)$

$$x = A(x^2+4) + (Bx+C)(x-1)$$

Put $x = 1$

$$1 = A(1^2+4) = \quad A = \frac{1}{5}$$

Compare coefficient of x^2 and constant

$$0 = A + B \Rightarrow \quad B = -A = -\frac{1}{5}$$

$$0 = 4A - C, \Rightarrow \quad C = 4A = \frac{4}{5}$$

$$I = \int \left[\frac{1}{5(x-1)} + \frac{-\frac{1}{5}x + \frac{4}{5}}{x^2+4}\right] dx$$

$$= \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{5}\int \frac{x-4}{x^2+4}\,dx$$

$$= \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{5\times 2}\int \frac{2x-8}{x^2+4}\,dx$$

$$= \frac{1}{5}\int \frac{dx}{x-1} - \frac{1}{10}\int \frac{2x\,dx}{x^2+4} + \frac{4}{5}\int \frac{dx}{x^2+4}$$

$$\boxed{\begin{aligned} x^2+4 &= t \\ 2x\,dx &= dt \end{aligned}}$$

$$= \cancel{\frac{1}{3}\int} \frac{1}{5}\log|x-1| - \frac{1}{10}\int \frac{dt}{t} + \frac{4}{5\cdot\times 2}\tan^{-1}\frac{x}{2} + c$$

$$= \frac{1}{5}\log|x-1| - \frac{1}{10}\log|t| + \frac{2}{5}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{5}\log|x-1| - \frac{1}{10}\log|x^2+4| + \frac{2}{5}\tan^{-1}x/2 + C$$

Type IV

Q4) $I = \int \frac{x^2}{(x^2+1)(x^2+4)} dx$

Let $x^2 = y$

$$\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4} \quad -(1)$$

Multiply both sides by $(y+1)(y+4)$

$$y = A(y+4) + B(y+1)$$

Put $y = -1$

$-1 = A(-1+4)$

$\Rightarrow A = \frac{-1}{3}$

Put $y = -4$

$-4 = B(-4+1)$

$B = \frac{4}{3}$

Put in (1)

$$\frac{y}{(y+1)(y+4)} = \frac{-1}{3(y+1)} + \frac{4}{3(y+4)}$$

Put value of $y = x^2$

$$I = \int \left[\frac{-1}{3(x^2+1)} + \frac{4}{3(x^2+4)} \right] dx$$

$$= \frac{-1}{3} \int \frac{dx}{x^2+1} + \frac{4}{3} \int \frac{dx}{x^2+4}$$

$$= \frac{-1}{3} \tan^{-1} x + \frac{4}{3 \times 2} \tan^{-1} \frac{x}{2} + c$$

$$= \frac{-1}{3} \tan^{-1} x + \frac{2}{3} \tan^{-1} \frac{x}{2} + c$$

Type V $\int \frac{x^2+1}{x^4+\lambda x^2+1}dx$, $\int \frac{x^2-1}{x^4+\lambda x^2+1}dx$, $\int \frac{dx}{x^4+\lambda x^2+1}$

where $\lambda \in R$.

1. Divide Nr. and Dr. by x^2
2. Express Dr. as $\left(x \pm \frac{1}{x}\right)^2 \pm k^2$
3. Put $x+\frac{1}{x}=t$ or $x-\frac{1}{x}=t$

Q.5 $I = \int \frac{x^2+1}{x^4+1}dx$

Divide Nr and Dr by x^2

$$= \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}dx$$

$$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+2-2}dx$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+(\sqrt{2})^2}dx$$

Let $x-\frac{1}{x}=t$

$\left(1+\frac{1}{x^2}\right)dx = dt$

$$= \int \frac{dt}{t^2+(\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}}+C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+C$$

Q.2 $I = \int \sqrt{\tan\theta}\, d\theta$

Let $\sqrt{\tan\theta} = x \Rightarrow \tan\theta = x^2$

$\sec^2\theta\, d\theta = 2x\, dx$

$d\theta = \frac{2x}{1+\tan^2\theta} dx$

$= \frac{2x}{1+x^4} dx$

$I = \int x \cdot \frac{2x}{1+x^4} dx$

$= \int \frac{2x^2}{1+x^4} dx$

$= \int \frac{x^2+x^2}{1+x^4} dx = \int \frac{x^2+1+x^2-1}{x^4+1} dx$

$= \int \frac{x^2+1}{x^4+1} dx + \int \frac{x^2-1}{x^4+1} dx$

$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx + \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx$ [Dividing Dr & Nr. by x^2]

$= \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}-2+2} dx + \int \frac{1-\frac{1}{x^2}}{\left(x^2+\frac{1}{x^2}+2-2\right)} dx$

$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+(\sqrt{2})^2} dx + \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-(\sqrt{2})^2} dx$

Put $x - \frac{1}{x} = t$ & $x + \frac{1}{x} = u$

$\left(1+\frac{1}{x^2}\right)dx = dt$; $\left(1-\frac{1}{x^2}\right)dx = du$

$I = \int \frac{dt}{t^2+(\sqrt{2})^2} + \int \frac{du}{u^2-(\sqrt{2})^2}$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log \left| \frac{u-\sqrt{2}}{u+\sqrt{2}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x - \frac{1}{x}}{\sqrt{2}} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}\, x} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| + C$$

Type VI $\int \frac{dx}{x(x^n+1)}$ where n is a natural number

$$I = \int \frac{dx}{x(x^n+1)}$$

Multiply Nr and Dr by x^{n-1}

$$I = \int \frac{x^{n-1}\, dx}{x^{n-1} x (x^n+1)}$$

$$= \int \frac{x^{n-1}\, dx}{x^n (x^n+1)}$$

Put $x^n = t \Rightarrow n x^{n-1} dx = dt$

$$I = \frac{1}{n} \int \frac{dt}{t(t+1)}$$

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$$I = \frac{1}{n} \int \frac{dt}{t} - \frac{1}{n} \int \frac{dt}{t+1}$$

$$= \frac{1}{n} \log|t| - \frac{1}{n} \log|t+1| + C$$

$$= \frac{1}{n} \log \left| \frac{t}{t+1} \right| + C$$

$$= \frac{1}{n} \log \left| \frac{x^n}{x^n+1} \right| + C$$