Integration (7)

Type 1 $\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Step 1 Reduce coefficient of x^2 to 1.

Step 2 Reduce the terms containing x into perfect square

Step 3 Integrate.

Q.1 $\int \frac{dx}{2x^2+x-1}$.

$$= \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}-\frac{1}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{\left[x^2+\frac{x}{2}+\left(\frac{1}{4}\right)^2\right]-\left(\frac{1}{4}\right)^2-\frac{1}{2}}$$ [Add and sub. $\left(\frac{1}{2}\text{ coeff of } x\right)^2$]

$$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2-\frac{9}{16}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot\frac{3}{4}}\log\left|\frac{x+\frac{1}{4}-\frac{3}{4}}{x+\frac{1}{4}+\frac{3}{4}}\right|+c$$ $\left\{\because \int \frac{dx}{x^2-a^2}=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c\right\}$

$$= \frac{1}{3}\log\left|\frac{4x-2}{4x+4}\right|+c$$

$$= \frac{1}{3}\log\left|\frac{2x-1}{2x+2}\right|+c$$

HW Q $\int \frac{dx}{3+2x-x^2}$

Hint $\int \frac{dx}{-[x^2-2x-3]} = -\int \frac{dx}{x^2-2x-3}$

Q.2

$$\int \frac{dx}{\sqrt{9+8x-x^2}}$$

$$= \int \frac{dx}{\sqrt{-[x^2-8x-9]}}$$

$$= \int \frac{dx}{\sqrt{-[x^2-8x+(4)^2-(4)^2-9]}}$$

$$= \int \frac{dx}{\sqrt{-[(x-4)^2-25]}}$$

$$= \int \frac{dx}{\sqrt{25-(x-4)^2}}$$

$$= \int \frac{dx}{\sqrt{(5)^2-(x-4)^2}}$$

$$= \sin^{-1}\frac{(x-4)}{5}+c \qquad \left\{\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+c\right\}$$

H.W $\int \frac{dx}{\sqrt{5x^2-2x}}$

Type II $\int \frac{\text{Linear}}{\text{Quadratic}}\,dx$ or $\int \frac{\text{Linear}}{\sqrt{\text{Quadratic}}}\,dx$

$$\int \frac{px+q}{ax^2+bx+c}\,dx \quad \text{or} \quad \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$$

Step 1 Put $ax^2+bx+c = t$

$(2ax+b)\,dx = dt$

Step 2 Reduce Nr. in terms of $2ax+b$

e.g. $I = \int \frac{px+q}{ax^2+bx+c}\,dx = p\int \frac{x+q/p}{ax^2+bx+c}\,dx$

$$= \frac{1}{2a}\int \frac{2ax+\frac{aq}{p}}{ax^2+bx+c}\,dx = \frac{1}{2a}\int \frac{(2ax+b)+\frac{aq}{p}-b}{(ax^2+bx+c)}\,dx$$

Q.3 $I = \int \frac{4x+1}{x^2+3x+2}\,dx$

Let $x^2+3x+2 = t$

$(2x+3)\,dx = dt$

$$I = 2\int \frac{2x+\frac{1}{2}}{x^2+3x+2}\,dx$$

$$= 2\int \frac{(2x+3)+\frac{1}{2}-3}{x^2+3x+2}\,dx$$

$$= 2\int \frac{2x+3}{x^2+3x+2}\,dx + 2\int \frac{-5/2}{x^2+3x+2}\,dx$$

$$= 2\int \frac{dt}{t} + 2\times -\frac{5}{2}\int \frac{dx}{x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+2}$$

$$= 2\int \frac{dt}{t} - 5\int \frac{dx}{\left(x+\frac{3}{2}\right)^2-\frac{1}{4}}$$

$$= 2\int \frac{dt}{t} - 5\int \frac{dx}{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

$$= 2\log t - 5\cdot\frac{1}{2\cdot\frac{1}{2}}\log\left|\frac{x+3/2-1/2}{x+3/2+1/2}\right| + C$$

$$= 2\log|x^2+3x+2| - 5\log\left|\frac{x+1}{x+2}\right| + C$$

H.W $\int \frac{x}{x^2+x+1}\,dx$, $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$

Q.4 Type III $\int \frac{dx}{a\sin^2 x + b\cos^2 x}$, $\int \frac{dx}{a+b\sin^2 x}$, $\int \frac{dx}{a+b\cos^2 x}$,

$\int \frac{dx}{(a\sin x + b\cos x)^2}$, $\int \frac{dx}{a+b\sin^2 x + c\cos^2 x}$

1. Divide both Nr. and Dr. by $\cos^2 x$
2. Reduce Dr. in terms of $\tan^2 x$
3. Put $\tan x = t$

Q.4 $\int \frac{dx}{4\cos^2 x + 9\sin^2 x}$

$$= \int \frac{\frac{1}{\cos^2 x}\,dx}{4\frac{\cos^2 x}{\cos^2 x} + 9\frac{\sin^2 x}{\cos^2 x}}$$

$$= \int \frac{\sec^2 x\,dx}{4 + 9\tan^2 x}$$

$$= \int \frac{dt}{4+9t^2}$$

Let $\tan x = t$

$\sec^2 x\,dx = dt$

$$= \frac{1}{9}\int \frac{dt}{t^2 + 4/9}$$

$$= \frac{1}{9}\int \frac{dt}{t^2 + (2/3)^2}$$

$$= \frac{1}{9} \cdot \frac{1}{2/3} \tan^{-1}\frac{3t}{2} + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3\tan x}{2}\right) + C$$

H.W $\int \frac{dx}{1+3\sin^2 x}$

Type IV $\int \frac{dx}{(a\sin x + b\cos x)}$, $\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a+b\cos x}$,

$\int \frac{dx}{(a\sin x + b\cos x + c)}$

(1) Put $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$, $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$

(2) Replace $1+\tan^2 x/2$ in the Nr. by $\sec^2 x/2$

(3) Put $\tan x/2 = t$

Q. 5 $I = \int \frac{dx}{5+4\cos x}$

$= \int \frac{dx}{5+4\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}$ $\left[\because \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right]$

$= \int \frac{\sec^2 x/2 \, dx}{5+5\tan^2 x/2 + 4 - 4\tan^2 x/2}$

$= \int \frac{\sec^2 x/2 \, dx}{\tan^2 x/2 + 9}$

$= \int \frac{\sec^2 x/2}{\tan^2 x/2 + (3)^2} dx$

$= 2\int \frac{dt}{t^2+(3)^2}$ Let $\tan x/2 = t$

$\frac{1}{2}\sec^2 x/2 \, dx = dt$

$= \frac{2}{3}\tan^{-1}\frac{t}{3} + C$

$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan x/2\right) + C$

H.W $\int \frac{dx}{1+\sin x+\cos x}$

Type V $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}\,dx$

(1.) Express $Nr = A\frac{d}{dx}(Dr) + B(Dr)$

(2) Integrate.

Q.6 $\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\,dx$

Let $2\sin x + 3\cos x = A\frac{d}{dx}(3\sin x + 4\cos x) +$
$B(3\sin x + 4\cos x)$

$= A(3\cos x - 4\sin x) + B(3\sin x + 4\cos x)$ — (I)

$= \cos x(3A + 4B) + \sin x(-4A + 3B)$

Compare coefficients of sinx and cosx

$2 = -4A + 3B$ — (1)
$3 = 3A + 4B$ — (2)

Multiply (1) by 3 and (2) by 4

$-12A + 9B = 6$
$12A + 16B = 12$

$25B = 18$
$B = \frac{18}{25}$

Put value of B in (1)

$2 = -4A + 3\times\frac{18}{25}$

$2 = -4A + \frac{54}{25}$

$2 - \frac{54}{25} = -4A$

$\frac{-4}{25\times -4} = A \Rightarrow A = \frac{1}{25}$

Put values of A & B in (I)

$2\sin x + 3\cos x = \frac{1}{25}(3\cos x - 4\sin x) + \frac{18}{25}(3\sin x + 4\cos x)$

$$I = \int \frac{\frac{1}{25}(3\cos x - 4\sin x) + \frac{18}{25}(3\sin x + 4\cos x)}{3\sin x + 4\cos x}\,dx$$

$$= \frac{1}{25}\int \frac{dt}{t} + \frac{18}{25}\int dx$$

$$= \frac{1}{25}\log|t| + \frac{18}{25}x + c$$

$$= \frac{1}{25}\log|3\sin x + 4\cos x| + \frac{18}{25}x + c$$

Type VI $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$

1) Express $Nr = A\frac{d}{dx}(Dr) + B(Dr) + C$

Q.7 $I = \int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\,dx$

$$3\cos x + 2 = A\frac{d}{dx}(\sin x + 2\cos x + 3) + B(\sin x + 2\cos x + 3) + C$$

$$= A(\cos x - 2\sin x + 0) + B(\sin x + 2\cos x + 3) + C \quad -(1)$$

$$= \cos x(A + 2B) + \sin x(-2A + B) + (0A + 3B + C)$$

Comparing coefficients of $\cos x$, $\sin x$, constants.

$3 = A + 2B \quad -(2)$

$0 = -2A + B \quad -(3) \Rightarrow B = 2A$

$2 = 0A + 3B + C \quad -(4)$

Put value from (3) in (2)

$3 = A + 4A \Rightarrow A = \frac{3}{5}$

$\Rightarrow B = 2A = \frac{6}{5}$

$C = 2 - 0A - 3B = \cancel{2 - \frac{9}{5} - \frac{18}{5} = \frac{2-27}{5}}$

$= \cancel{\frac{10}{5} - \frac{17}{5}}$ $2 - \frac{18}{5} = \frac{-8}{5}$

Put the values of A, B, C in (1)

$$3\cos x + 2 = \frac{3}{5}(\cos x - 2\sin x) + \frac{6}{5}(\sin x + 2\cos x + 3) - \frac{8}{5}$$

$$I = \int \frac{\frac{3}{5}(\cos x - 2\sin x) + \frac{6}{5}(\sin x + 2\cos x + 3) - \frac{8}{5}}{\sin x + 2\cos x + 3}\,dx$$

$$= \frac{3}{5}\int \frac{dp}{p} + \frac{6}{5}\int dx - \frac{8}{5}\int \frac{dx}{\sin x + 2\cos x + 3}$$

where $p = \sin x + 2\cos x + 3$

$$= \frac{3}{5}\log|p| + \frac{6}{5}x - \frac{8}{5}\int \frac{dx}{\frac{2\tan x/2}{1+\tan^2 x/2} + 2\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right) + 3}$$

$$= \frac{3}{5}\log|p| + \frac{6}{5}x - \frac{8}{5}\int \frac{\sec^2 x/2\,dx}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3 + 3\tan^2 x/2}$$

$$= \frac{3}{5}\log|\sin x + 2\cos x + 3| + \frac{6}{5}x - \frac{8}{5}\int \frac{\sec^2 x/2\,dx}{\tan^2 x/2 + 2\tan x/2 + 5}$$

$$= \frac{3}{5}\log|\sin x + 2\cos x + 3| + \frac{6}{5}x - \frac{8}{5}\times 2\int \frac{dt}{t^2 + 2t + 5}$$

Let $\tan x/2 = t$

$\frac{1}{2}\sec^2 x/2\,dx = dt$

$$= \frac{3}{5}\log|\sin x + 2\cos x + 3| + \frac{6}{5}x - \frac{16}{5}\int \frac{dt}{(t^2 + 2t + 1) + 4}$$

$$= \frac{3}{5}\log|\sin x + 2\cos x + 3| + \frac{6}{5}x - \frac{16}{5}\int \frac{dt}{(t+1)^2 + 4}$$

$$= \frac{3}{5}\log|\sin x + 2\cos x + 3| + \frac{6}{5}x - \frac{16}{5}\times\frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right) + C$$ ← Put value of t