Integration (8)

Integration By Parts

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

Here u = first function

v = second function

Choose the function that comes first in ILATE as first function.

I – Inverse trigonometric function

L – Logarithmic function

A – Algebraic function

T – Trigonometric function

E – Exponential function.

Q.1 $I = \int \underset{(I)}{x}\ \underset{(II)}{\sin 3x}\,dx$

$= x\int \sin 3x\,dx - \int\left\{\frac{dx}{dx}\cdot\int \sin 3x\,dx\right\}dx$

x = algebraic fn. = I^{st} fn.
$\sin 3x$ = trigonometric fn. = II^{nd} fn.

$= x\left(\frac{-\cos 3x}{3}\right) - \int 1\cdot\left(-\frac{\cos 3x}{3}\right)dx$

$= -\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + C$

Q.2 Rem $\int \log x\,dx = \int \underset{II}{1}\cdot\underset{I}{\log x}\,dx$

$= \log x\int 1\,dx - \int\left\{\left(\frac{d}{dx}\log x\right)\int 1\,dx\right\}dx$

$= x\log x - \int \frac{1}{x}\cdot x\,dx$

$= x\log x - x + C$

$1 = x^0$ = algebraic fn.

$\log x$ = logarithmic fn. = I^{st} fn.

Q.3 $I = \int \underset{I}{x^2} \underset{II}{e^x} dx$

$= x^2 \int e^x dx - \int (2\underset{I}{x} \cdot \underset{II}{e^x}) dx$

$= x^2 e^x - 2\left[x \cdot \int e^x dx - \int 1 \cdot e^x dx\right]$

$= x^2 e^x - 2xe^x + 2e^x + c$

$= x^2 e^x - 2[xe^x + e^x] + c$

Q.4 $I = \int \tan^{-1}x = \int \underset{(II)}{1} \cdot \underset{(I)}{\tan^{-1}x}$

$= \tan^{-1}x \int dx - \int \left(\frac{d}{dx}\tan^{-1}x\right) \cdot x\, dx$

$= \tan^{-1}x \cdot x - \int \frac{x}{1+x^2} dx$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{dt}{t}$

$= x\tan^{-1}x - \frac{1}{2}\log|t| + c.$

$\boxed{\begin{matrix} 1+x^2 = t \\ 2x\,dx = dt \end{matrix}}$

$= x\tan^{-1}x - \frac{1}{2}\log|1+x^2| + c$

Q.5 Imp $I = \int \sec^3 x\, dx = \int \underset{(I)}{\sec x} \cdot \underset{(II)}{\sec^2 x}\, dx$

(II is easily integrable)

$= \sec x \int \sec^2 x\, dx - \int \left(\frac{d}{dx}\sec x\right) \cdot \tan x\, dx$

$= \sec x \cdot \tan x - \int \sec x\, \underline{\tan x \cdot \tan x}\, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$

$= \sec x \tan x - \int \sec^3 x - \int \sec x\, dx$

$= \sec x \tan x - I - \log|\sec x + \tan x| + c$

$2I = \sec x \tan x - \log|\sec x + \tan x| + c$

$I = \frac{1}{2}\left[\sec x \tan x + \log|\sec x + \tan x|\right] + c$

H.W

Q. $\int x^2 \cos x \, dx$, $\int \sin^{-1} x \, dx$

Type II $\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + c$

Proof $\int e^x \{f(x) + f'(x)\} dx$

$= \int \underset{II}{e^x} \underset{I}{f(x)} dx + \int e^x f'(x) dx$

$= f(x) e^x - \cancel{\int f'(x) e^x dx} + \cancel{\int e^x f'(x) dx} + c$

$= f(x) e^x + c$

Q.6 $I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$

Let $f(x) = \frac{1}{x}$

$\Rightarrow f'(x) = \frac{-1}{x^2}$

$\therefore I = \int e^x (f(x) + f'(x)) dx$

$= e^x f(x) + c$

$= \frac{e^x}{x} + c$

H.W $\int e^x (\sin x + \cos x) dx$

Q.7 $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right)dx$

$= \int e^x \left(\frac{1-2\sin x/_2 \cos x/_2}{2\sin^2 x/_2}\right)dx$

$= \int e^x \left(\frac{1}{2}\operatorname{cosec}^2 x/_2 - \cot x/_2\right)dx$

Let $\cot x/_2 = f(x)$

$-\frac{1}{2}\operatorname{cosec}^2 x/_2 = f'(x)$

$= -\int e^x \left(\cot x/_2 - \frac{1}{2}\operatorname{cosec}^2 x/_2\right)dx$

$= -\int e^x (f(x) + f'(x))dx$

$= -e^x f(x) + c$

$= -e^x \cot x/_2 + c$

Type III $\int e^{ax}\sin bx\,dx$ or $\int e^{ax}\cos bx\,dx$

* $\rightarrow$ Use integration by parts.

Q.8 $I = \int \underset{\text{II}}{e^{ax}} \underset{\text{I}}{\sin bx}\,dx$

$= \sin bx \cdot \frac{e^{ax}}{a} - \int b\overset{\text{I}}{\cos bx} \cdot \overset{\text{II}}{\frac{e^{ax}}{a}}dx$

$= \sin bx \cdot \frac{e^{ax}}{a} - \frac{b}{a}\left[\cos bx \cdot \frac{e^{ax}}{a} + \int b\sin bx \frac{e^{ax}}{a}dx\right]$

$= \sin bx \cdot \frac{e^{ax}}{a} - \frac{b}{a^2}\cos bx\, e^{ax} - \frac{b^2}{a^2} I$

$$I + \frac{b^2}{a^2} I = e^{ax}\left[\frac{\sin bx}{a} - \frac{b\cos bx}{a^2}\right]$$

$$\frac{a^2+b^2}{a^2} I = e^{ax}\left[\frac{a\sin bx - b\cos bx}{a^2}\right]$$

$$I = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right] + C$$

H.W Prove

Q. $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] + C$

Q. $\int e^{x}\sin^2 x\, dx$

Hint $\int e^{x}\sin^2 x\, dx = \int e^{x}\left(\frac{1-\cos 2x}{2}\right)dx$

$= \frac{1}{2}\int \left(e^{x} - e^{x}\cos 2x\right) dx$

Some Important Integrals

(1) Rem $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2} x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

$$I = \int \underset{II}{1} \cdot \underset{I}{\sqrt{a^2-x^2}}\,dx$$

$$= \sqrt{a^2-x^2}\cdot x - \int \frac{1}{2\sqrt{a^2-x^2}}(-2x)\cdot x\,dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2+a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx - a^2\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$= x\sqrt{a^2-x^2} - I - a^2\sin^{-1}\frac{x}{a} + C$$

$$2I = x\sqrt{a^2-x^2} - a^2\sin^{-1}\frac{x}{a} + C$$

$$I = \frac{x}{2}\sqrt{a^2-x^2} - \frac{a^2}{2}\sin^{-1}x/a + C$$

(2) $\int\sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| + C$

$$I = \int \underset{(II)}{1} \cdot \underset{(I)}{\sqrt{a^2+x^2}}\,dx$$

$$= \sqrt{a^2+x^2}\cdot x - \int \frac{1}{2\sqrt{a^2+x^2}}\cdot 2x\cdot x\,dx$$

$$= x\sqrt{a^2+x^2} - \int \frac{x^2}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - \int \frac{a^2+x^2-a^2}{\sqrt{a^2+x^2}}\,dx$$

$= x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,dx + a^2 \int \frac{dx}{\sqrt{a^2+x^2}}$

$= x\sqrt{a^2+x^2} - I + a^2 \log|x+\sqrt{a^2+x^2}| + C$

$2I = x\sqrt{a^2+x^2} + a^2 \log|x+\sqrt{a^2+x^2}| + C$

$I = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log|x+\sqrt{a^2+x^2}| + C$

H.W Prove: $\boxed{\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C}$

Type 1 $\int \sqrt{ax^2+bx+c}\,dx$

step 1 - Make coefficient of x^2 unity (1).

step 2 - Reduce terms containing x^2 & x into perfect square.

Q. $\int \sqrt{x^2+2x+5}\,dx$

$= \int \sqrt{x^2+2x+1+4}\,dx$

$= \int \sqrt{(x+1)^2+(2)^2}\,dx$

$= \frac{(x+1)}{2}\sqrt{x^2+2x+5} + \frac{\cancel{4}^{2}}{\cancel{2}}\log|x+1+\sqrt{x^2+2x+5}| + C$