Integration (5)

Type 1 Q.1 $I = \int \dfrac{dx}{\sin(x-a)\sin(x-b)}$

* Multiply & divide by $\sin(a-b)$

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)\,dx}{\sin(x-a)\sin(x-b)}$$

Now $a - b = x - x + a - b$

$= (x-b) - (x-a)$

$\sin(a-b) = \sin[(\overset{A}{x-b}) - (\overset{B}{x-a})]$

$= \sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)$

$$I = \frac{1}{\sin(a-b)} \int \frac{[\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)]\,dx}{\sin(x-a)\sin(x-b)}$$

$$= \frac{1}{\sin(a-b)} \int [\cot(x-a) - \cot(x-b)]\,dx$$

$$= \frac{1}{\sin(a-b)} \left[\log\sin(x-a) - \log\sin(x-b)\right] + C$$

$$= \frac{1}{\sin(a-b)} \log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$$

H.W (1) $\displaystyle\int \frac{dx}{\cos(x-a)\cos(x-b)}$

[Hint : Multiply and divide by $\sin(a-b)$]

(2) $\displaystyle\int \frac{dx}{\sin(x-a)\cos(x-b)}$

[Hint : Multiply & divide by $\cos(a-b)$]

Type 2 : $\int \sin^m x$, $\int \cos^m x\,dx$, where $m \le 4$

Use the following identities

(i) $\sin^2 x = \frac{1-\cos 2x}{2}$ (ii) $\cos^2 x = \frac{1+\cos 2x}{2}$

(iii) $\sin 3x = 3\sin x - 4\sin^3 x$

(iv) $\cos 3x = 4\cos^3 x - 3\cos x$

Q.2 $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$

H.W $\int \cos^2 x\,dx$

Q 3 $\int \cos^3 x\,dx = \int \left(\frac{3\cos x + \cos 3x}{4}\right) dx$ $\left\{ \because 4\cos^3 x = \cos 3x + 3\cos x \right\}$

$= \frac{1}{4}\left\{3\sin x + \frac{\sin 3x}{3}\right\} + c$

H.W $\int \sin^3 x\,dx$

Q 4 $I = \int \sin^3 x \cos^3 x\,dx$

$= \int (\sin x \cos x)^3\,dx$

$= \int \left(\frac{2\sin x\cos x}{2}\right)^3 dx$

$= \frac{1}{8}\int (\sin 2x)^3\,dx = \frac{1}{8}\int \sin^3 2x\,dx$

$= \frac{1}{8}\int \frac{3\sin 2x - \sin 6x}{4}\,dx$

$= \frac{1}{32}\left\{-\frac{3\cos 2x}{2} + \frac{\cos 6x}{6}\right\} + c$

H.W $\int \sin^2 x \cos^2 x \, dx$

Q.5 $\int \sin^4 x \, dx$

$= \int (\sin^2 x)^2 dx$

$= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$

$= \frac{1}{4}\int [1 - 2\cos 2x + \cos^2 2x] dx$

$= \frac{1}{4}\int \left[1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right] dx$

$= \frac{1}{4}\int \left[1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right] dx$

$= \frac{1}{4}\int \left[\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right] dx$

$= \frac{1}{8}\int [3 - 4\cos 2x + \cos 4x] dx$

$= \frac{1}{8}\left[3x - 4\frac{\sin 2x}{2} + \frac{\sin 4x}{4}\right] + c$

$= \frac{1}{8}\left[3x - 2\sin 2x + \frac{\sin 4x}{4}\right] + c$

H.W $\int \cos^4 x \, dx$

Type 3 : $\int \sin^m x \cos^n x\, dx$; $m, n \in N$.

Note

① If the power of $\sin x$ is an odd +ve integer put $\cos x = t$

② If the power of $\cos x$ is an odd +ve integer put $\sin x = t$

③ If the power of $\cos x$ & $\sin x$ both are odd +ve integers put either $\sin x = t$ or $\cos x = t$

④ If the power of both $\sin x$ and $\cos x$ are even +ve integers

Use trigonometric identities like

$\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$

Q.6 $I = \int \sin^3 x \cos^4 x\, dx$

$= \int \sin^2 x \cos^4 x \sin x\, dx$ Let $\cos x = t$

$-\sin x\, dx = dt$

$= \int (1-\cos^2 x) \cos^4 x \sin x\, dx$

$= -\int (1-t^2) . t^4\, dt$

$= -\int (t^4 - t^6)\, dt$

$= -\left[\frac{t^5}{5} - \frac{t^7}{7}\right] + c$

$= \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + c$

H.W $\int \sin^2 x \cos^5 x\, dx$

Type 4 $\int \sin^m x \cos^n x \, dx$; $m, n \in Q$ s.t.

$m+n$ is a negative even integer.

Note: (1) Divide Nr. and Dr. by $\cos^k x$ where $k = -(m+n)$.

(2) Reduce the integrand in the form of $\tan x$ and $\sec^2 x$.

(3) Put $\tan x = t$

Q.7 $\int \frac{\sin^4 x}{\cos^8 x} dx$

Here $m = 4$; $n = -8$ $\Rightarrow m+n = -4$ (-ve even integer)

Divide Nr. and Dr. by $\cos^4 x$

$$I = \int \frac{\frac{\sin^4 x}{\cos^4 x}}{\frac{\cos^8 x}{\cos^4 x}} dx$$

$$= \int \frac{\tan^4 x}{\cos^4 x} dx$$

$$= \int \tan^4 x \sec^2 x . \sec^2 x \, dx$$

$$= \int \tan^4 x (1+\tan^2 x) \sec^2 x \, dx$$

Let $\tan x = t$

$\sec^2 x \, dx = dt$

$$= \int t^4 (1+t^2) \, dt$$

$$= \int (t^4 + t^6) \, dt$$

$$= \frac{t^5}{5} + \frac{t^7}{7} + C = \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$$

Q.8 $I = \int \frac{1}{\sin^3 x \cos x} dx$

$m = -3 ; n = -1 \Rightarrow m+n = -4$

$\therefore$ Divide Nr. and Dr. by $\cos^4 x$

$I = \int \frac{\frac{1}{\cos^4 x}}{\frac{\sin^3 x \cos x}{\cos^4 x}} dx$

$= \int \frac{\sec^2 x \sec^2 x \, dx}{\tan^3 x}$

$= \int \frac{1+\tan^2 x}{\tan^3 x} \sec^2 x \, dx$

$\tan x = t$

$\sec^2 x \, dx = dt$

$= \int \frac{(1+t^2)}{t^3} dt$

$= \int \left[\frac{1}{t^3} + \frac{t^2}{t^3}\right] dt$

$= \frac{t^{-2}}{-2} + \log|t| + c$

$= \frac{-1}{2t^2} + \log|t| + c$

$= \frac{-1}{2(\tan x)^2} + \log|\tan x| + c$