Integration (6)

Some Special Integrals

(1) $I = \int \frac{dx}{x^2+a^2}$

Let $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$$I = \int \frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2}$$

$$= \int \frac{a\sec^2\theta}{a^2\sec^2\theta}\, d\theta$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Rem $\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C}$

(2) $I = \int \frac{dx}{x^2-a^2}$

$$\frac{1}{x^2-a^2} = \frac{1}{2a}\left\{\frac{1}{x-a} - \frac{1}{x+a}\right\}$$ {Partial fractions Rule of everywhere not here}

$$I = \frac{1}{2a}\int \left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx$$

$$= \frac{1}{2a}\left[\log|x-a| - \log|x+a|\right] + C$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

Rem $\boxed{\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C}$

(3) $I = \int \frac{1}{a^2 - x^2} dx$

$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left\{\frac{1}{a+x} + \frac{1}{a-x}\right\}$ (Partial fractions)

$I = \frac{1}{2a}\left[\int \frac{1}{a+x} dx - \int \frac{1}{a-x} dx\right]$

$= \frac{1}{2a}\left[\log|a+x| - \log|a-x|\right] + c$

$= \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

Form $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

(4) $I = \int \frac{dx}{\sqrt{a^2 - x^2}}$

Let $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}}$

$= \int \frac{a\cos\theta}{a\sqrt{1 - \sin^2\theta}} d\theta$

$= \int \frac{a\cos\theta}{a\cos\theta} d\theta$

$= \int d\theta$

$= \theta + c$

$= \sin^{-1}\frac{x}{a} + c$

Form $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

(5) $I = \int \frac{dx}{\sqrt{a^2+x^2}}$

Let $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$$I = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2+a^2\tan^2\theta}}$$

$$= \int \frac{a\sec^2\theta}{a\sqrt{1+\tan^2\theta}}\, d\theta$$

$$= \int \frac{\sec^2\theta}{\sec\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \log|\sec\theta+\tan\theta| + C$$

$$= \log\left|\sqrt{(1+\tan^2\theta)}+\tan\theta\right| + C$$

$$= \log\left|\sqrt{\left(1+\frac{x^2}{a^2}\right)}+\frac{x}{a}\right| + C$$

$$= \log\left|\sqrt{(a^2+x^2)}+x\right| - \log a + C$$

$$= \log\left|\sqrt{a^2+x^2}+x\right| + C'$$

$[C - \log a = \text{constant} = C']$

Rem $\int \frac{dx}{\sqrt{a^2+x^2}} = \log\left|x+\sqrt{a^2+x^2}\right| + C$

(6) $I = \int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}| + C$

Let $x = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

$$I = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}\, d\theta$$

$$= \int \frac{a\sec\theta\tan\theta}{a\sqrt{\sec^2\theta-1}}\, d\theta$$

$$= \int \frac{\sec\theta\tan\theta}{\tan\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \log|\sec\theta+\tan\theta| + C$$

$$= \log|\sec\theta+\sqrt{\sec^2\theta-1}| + C$$

$$= \log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right| + C$$

$$= \log|x+\sqrt{x^2-a^2}| - \log a + C$$

$$= \log|x+\sqrt{x^2-a^2}| + c'$$

$$[c' = c - \log a = \text{Constant}]$$

Rem $\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}| + C$

Q.1 $\int \frac{dx}{4+9x^2}$

$= \frac{1}{9}\int \frac{dx}{x^2+\frac{4}{9}}$ [Make the coefficient of x^2 one]

$= \frac{1}{9}\int \frac{dx}{x^2+\left(\frac{2}{3}\right)^2}$

$= \frac{1}{9\times\frac{2}{3}} \tan^{-1}\frac{x}{\frac{2}{3}} + C$ $\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$

$= \frac{1}{6}\tan^{-1}\frac{3x}{2} + C$

Q.2 $\int \frac{dx}{16-9x^2}$

$= \frac{1}{9}\int \frac{dx}{\frac{16}{9}-x^2}$ [Make coefficient of x^2 +1 or -1]

$= \frac{1}{9}\int \frac{dx}{\left(\frac{4}{3}\right)^2-x^2}$

$= \frac{1}{9}\cdot\frac{1}{2\times\frac{4}{3}} \log\left|\frac{\frac{4}{3}+x}{\frac{4}{3}-x}\right| + C$ $\left\{\because \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\right\}$

$= \frac{1}{24}\log\left|\frac{4+3x}{4-3x}\right| + C$

Q.3 $\int \frac{dx}{\sqrt{9-25x^2}}$

$= \frac{1}{\sqrt{25}} \int \frac{dx}{\sqrt{\frac{9}{25}-x^2}}$ [Make coeff of x^2 either +1 or -1]

$= \frac{1}{5} \int \frac{dx}{\sqrt{\left(\frac{3}{5}\right)^2 - x^2}}$

$= \frac{1}{5} \cdot \sin^{-1} \frac{x}{3/5} + C$ $\left\{ \because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C \right\}$

$= \frac{1}{5} \sin^{-1} \frac{5x}{3} + C$

Q.4 $I = \int \frac{dx}{\sqrt{(2-x)^2+1}}$

$= -\log \left| (2-x) + \sqrt{(2-x)^2+1} \right| + C$

$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log \left| x + \sqrt{x^2+a^2} \right| + C \right.$